Section 10.5 Area and Arc Length in Polar Coordinates

1. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{\pi/2}^{\pi} (2\sin\theta)^2\,d\theta$

$= 2\int_{\pi/2}^{\pi} \sin^2\theta\,d\theta$

2. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{3\pi/4}^{5\pi/4} (\cos 2\theta)^2\,d\theta$

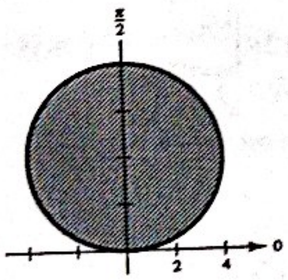
3. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{\pi/2}^{3\pi/2} (1-\sin\theta)^2\,d\theta$

4. $A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta$

$= \frac{1}{2}\int_{0}^{\pi} (1-\cos 2\theta)^2\,d\theta$

5. (a) $r = 8\sin\theta$



$A = \pi(4)^2 = 16\pi$

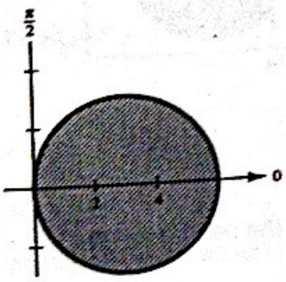
(b) $A = 2\left(\frac{1}{2}\right)\int_{0}^{\pi/2} [8\sin\theta]^2\,d\theta$

$= 64\int_{0}^{\pi/2} \sin^2\theta\,d\theta$

$= 32\int_{0}^{\pi/2} (1-\cos 2\theta)\,d\theta$

$= 32\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 16\pi$

6. (a) $r = 3\cos\theta$



$A = \pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$

(b) $A = 2\left(\frac{1}{2}\right)\int_{0}^{\pi/2} [3\cos\theta]^2\,d\theta$

$= 9\int_{0}^{\pi/2} \cos^2\theta\,d\theta$

$= \frac{9}{2}\int_{0}^{\pi/2} (1+\cos 2\theta)\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{9\pi}{4}$

7. $A = 2\left[\frac{1}{2}\int_{0}^{\pi/6} (2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{3}$

8. $A = 2\left[\frac{1}{2}\int_{0}^{\pi/4} (6\sin 2\theta)^2\,d\theta\right] = 36\int_{0}^{\pi/4} \sin^2 2\theta\,d\theta$

$= 36\int_{0}^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta$

$= 18\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= 18\left[\frac{\pi}{4}\right] = \frac{9\pi}{2}$

9. $A = 2\left[\frac{1}{2}\int_0^{\pi/4} (\cos 2\theta)^2\, d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}$

10. $A = 2\left[\frac{1}{2}\int_0^{\pi/10} (\cos 5\theta)^2\, d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

11. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

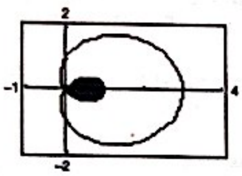
$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

12. $A = 2\left[\frac{1}{2}\int_0^{\pi/2} (1 - \sin\theta)^2\, d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

13. $A = 2\frac{1}{2}\left[\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= \left[3\theta + 4\sin\theta + \sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$

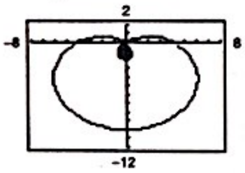


14. $A = 2\left[\frac{1}{2}\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right]$

$= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48\sin\theta + 36\sin^2\theta]\, d\theta$

$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$

$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$

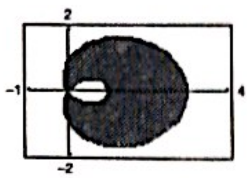


15. The area inside the outer loop is

$2\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$

From the result of Exercise 13, the area between the loops is

$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$



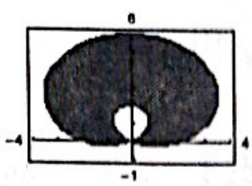
16. Four times the area in Exercise 15, $A = 4(\pi + 3\sqrt{3})$. More specifically, we see that the area inside the outer loop is

$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = \int_{-\pi/6}^{\pi/2} (4 + 16\sin\theta + 16\sin^2\theta)\, d\theta = 8\pi + 6\sqrt{3}.$

The area inside the inner loop is

$2\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2\sin\theta))^2\, d\theta\right] = 4\pi - 6\sqrt{3}.$

Thus, the area between the loops is $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}.$

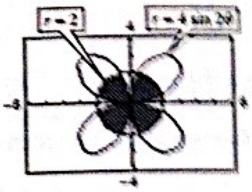


31. From Exercise 25, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12}(4\sin 2\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12}(2)^2\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2}(4\sin 2\theta)^2\,d\theta$$

$$= 16\int_0^{\pi/12}\sin^2(2\theta)\,d\theta + 2\int_{\pi/12}^{5\pi/12}d\theta \text{ (by symmetry of the petal)}$$

$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

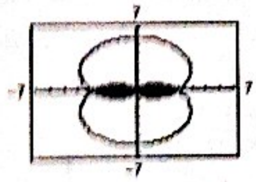


$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}(4\pi - 3\sqrt{3})$$

32. $$A = 4\left[\frac{1}{2}\int_0^{\pi/2} 9(1 - \sin\theta)^2\,d\theta\right]$$

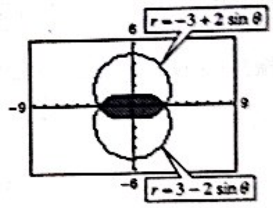
$$= 18\int_0^{\pi/2}(1 - \sin\theta)^2\,d\theta = \frac{9}{2}(3\pi - 8)$$

(from Exercise 14)



33. $$A = 4\left[\frac{1}{2}\int_0^{\pi/2}(3 - 2\sin\theta)^2\,d\theta\right]$$

$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$



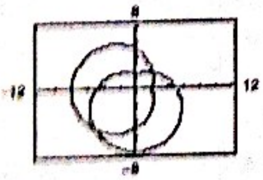
34. $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4}(5 - 3\sin\theta)^2\,d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

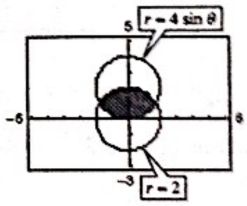
$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$



35. $$A = 2\left[\frac{1}{2}\int_0^{\pi/6}(4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}(2)^2\,d\theta\right]$$

$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

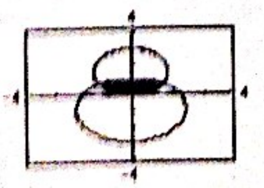
$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}(4\pi - 3\sqrt{3})$$



36. $$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2}(2 - \sin\theta)^2\,d\theta\right]$$

$$= \int_{\pi/6}^{\pi/2}(-4\cos 2\theta + 4\sin\theta)\,d\theta$$

$$= \left[-2\sin(2\theta) - 4\cos\theta\right]_{\pi/6}^{\pi/2} = 3\sqrt{3}$$



37. $$A = 2\left[\frac{1}{2}\int_0^{\pi}[a(1 + \cos\theta)]^2\,d\theta\right] - \frac{a^2\pi}{4}$$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$

$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$